Exploring Possible Choices of the Tikhonov Regularization Parameter for the Method of Fundamental Solutions in Electrocardiography

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Abstract

The inverse problem of electrocardiographic imaging (ECGI), i.e. computing epicardial potentials from the body surface measured potentials, is a challenging problem. In this setting, Tikhonov regularization is commonly employed, weighted by a regularization parameter. This parameter has an important influence on the solution.

In this work, we show the feasibility of two methods to choose the regularization parameter when using the method of fundamental solution, or MFS (a homogeneous meshless scheme based). These methods are i) a novel automatic technique based on the Discrete Picard condition (DPC), which we named ADPC and ii) the U-curve method introduced in other fields for cases where the well-known L-curve method fails or over-regularize the solution. We calculated the Tikhonov solution with the ADPC and U-curve methods for experimental data from the free distributed Experimental Data and Geometric Analysis Repository (EDGAR), and we compared them to the solution obtained with CRESO and L-curve procedures that are the two extensively used techniques in the ECGI.

1. Introduction

The electrocardiographic imaging (ECGI) inverse problem of computing epicardial potentials, Φ_E, from the body surface measured potentials, Φ_T, is an ill-posed problem, needing regularization to yield realistic and unique solution [1]. And different methods have been proposed with this end [1-4].

In [3], the authors studied the performance of fourteen algorithms for complex propagation patterns, and they concluded that (without prior information about the Φ_E), the simple two-norm Tikhonov method, may provide a similar solution than other more sophisticated techniques. Similarly, the authors in [4] studied the performance of thirteen reconstruction algorithms, concluding that on average little differences were found among the three main groups of techniques considered (i.e. Tikhonov, iterative methods, and non-quadratique techniques). Therefore, Tikhonov regularization method seems to be the preferred technique to solve the ECGI inverse problem.

In Tikhonov scheme, the regularization parameter has to be determined to find a balance between solutions purely based on the Φ_T and solutions too strictly constrained.

In this work, we will focus on the two-norm Tikhonov regularization technique for the MFS, a homogeneous meshless method adapted to ECGI in [5]. Specifically, we will focus on the choice of the regularization parameter.

In [6] we showed that the influence of the regularization parameter can decrease by optimizing the transfer matrix used (e.g. decreasing its ill-conditioning by adjusting some of its key elements [6]). This result links with the conclusion of [3], where the authors stated that no major difference was found by changing the regularization parameters, but show also notably degradation of the reconstruction performance, when errors were introduced in the transfer matrix. However, the transfer matrix depends on the conductivities (constant if considering homogeneity), the boundary conditions, and the geometries involved (and in the case of the MFS matrix depends also on the sources locations). Since errors such as the ones occurring in the segmentation process are not always perceived, suitable parameter choice techniques for each particular problem are necessary. Finally, different kind of data patterns may require different parameter choices.

The automatic regularization parameter choice method previously used in the MFS ECGI, is the Composite Residual and Smoothing Operator (CRESO) technique [5]. And when other numerical methods were used in the ECGI setting (such as the Boundary/Finite Element method), the L-curve criterion has been highly used by the community [2-4]. However, is well-known that the behavior of the

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parameter choice methods is problem-dependent [7]. While both methods have been studied in the inverse problems literature [2-5, 7], the L-curve method showed lack of: robustness dealing with large-scale problems [3], convergence (in particular applications) [8,9], and efficacy (over-smoothing the solution). Then, methods such as the U-curve [10] have been introduced to overcome these problems [9]. In addition, Discrete Picard Condition (DPC) [7] is often used to check the suitability of a chosen regularization parameter for Tikhonov.

The overarching goal of this paper is to show the feasibility of the U-curve method, and of a new automatic method, ADPC, based on the DPC [11]. To this end, we used experimental data from the free distributed Experimental Data and Geometric Analysis Repository, or EDGAR [12], an Internet-based archive of curated data freely distributed.

This proceeding is organized as follows: First, we present the MFS and Tikhonov regularization, we give more details about DPC and its role, and we describe the different regularization parameter choice techniques (existent ones and new ones). Secondly, the experimental data used, as well as the statistical analysis completed to compare the results are introduced. Thirdly, we summarize the main results obtained. And finally, we discuss the issues raised and draw conclusions.

2. Methods

2.1. Method of fundamental solutions and Tikhonov regularization

MFS was adapted to ECGI to overcome some of the issues of the classical meshes-based methods [5]. It does not need the topological relations between nodes and so completely avoids disadvantages of accuracy degradation and complexity augmentation frequently encountered in classical numerical methods because of remeshing.

In the MFS, the potentials are expressed as a linear combination of the Laplace fundamental solution over a discrete set of virtual source points placed outside of the domain of interest, Ω, where Ω is the volume conductor enclosed by the epicardial surface, Γ_E, and the body surface, Γ_T. Specifically, the potential Φ for $x \in \Omega$ is sought as

$$\Phi(x) = a_0 + \sum_{j=1}^{N_S} f(x - y_j) a_j, \tag{1}$$

where the $(y_j)_{j=1..N_S}$ are the N_S locations of the sources $(y_j \notin \Omega)$, and the $(a_j)_{j=1..N_S}$ are their coefficients. Here, f stands for the fundamental solution to the Laplace equation, $f(x, y_j) = \frac{1}{4\pi}\frac{1}{|x-y_j|}$, where $|x - y_j|$ is the 3D Euclidean distance. And the $N_S = N_T + N_E$ virtual sources locations are fixed by deflating the $(x_i^E)_{i=1,2,\cdots,N_E}$ locations at Γ_E and inflating the $(x_i^T)_{i=1,2,\cdots,N_T}$ electrodes locations at Γ_T. Both, inflation and deflation relative to the geometrical center of the heart, such as in [5].

The sought $\Phi_E = \left(\Phi(x_i^E)\right)_{i=1,\cdots,N_E}$ potentials on Γ_E can be defined by (1) as

$$\Phi(x_i^E) = a_0 + \sum_{j=1}^{N_S} f\left(x_i^E - y_j\right) a_j, \tag{2}$$

where the only unknowns are the sources coefficients.

To find these coefficients $(a_0, a_1, \cdots, a_{N_S})$ the Dirichlet $(\Phi = \Phi_T)$ and the homogeneous Neumann or zero-flux $(\partial_n \Phi = 0)$ boundary conditions on Γ_T are imposed in an equivalent manner by means of the potential definition (1) and its normal derivatives. This yields to the linear system

$$\Phi(x_i^T) = a_0 + \sum_{j=1}^{N_S} f\left(x_i^T - y_j\right) a_j = \Phi_T,$$
$$\partial_n \Phi(x_i^T) = a_0 + \sum_{j=1}^{N_S} \partial_{n_i} f\left(x_i^T - y_j\right) a_j = 0, \tag{3}$$

where $\Phi_T = (\Phi_i)_{i=1,\cdots,N_T}$ are the potentials recorded on the $(x_i^T)_{i=1,2,\cdots,N_T}$ torso electrodes locations. And (3) can be written in a matricial notation as $Ma = b$, where the sources coefficients, $a \epsilon \mathbb{R}^{1+N_S}$ are found by minimizing

$$J(a, \alpha) = \|Ma - b\|^2 + \alpha \|a\|^2, \tag{4}$$

being $b = \binom{\Phi_T^*}{0}$ a $2N_T x 1$ vector, $\Phi_T^* = (\Phi_i^*)_{i=1,\cdots,N_T}$ the potentials recorded on N_T torso electrodes, and $\alpha > 0$ the Tikhonov regularization parameter. α controls the balance between the residual norm, $\|Ma - b\|^2$(i.e. the accuracy of the sources coefficients a predicting the given boundary conditions at Γ_T), and the regularized solution norm, $\|a\|^2$.

By equalling the gradient of (4) to zero and doing the SVD of $M = USV^T$, the solution of (4) can be written as

$$a_\alpha = \sum_{i=1}^{r} \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2} \frac{u_i^T b}{\sigma_i} v_i, \tag{5}$$

being $r = \min(2N_T, N_{S+1})$ and $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_r > 0$ the singular values (SVs), or diagonal values of S.

2.2. Discrete Picard Condition (DPC)

The DPC determines how well the regularized solution approximates the unknown, exact solution. The DPC is satisfied if the data space coefficients $|u_i^T b|$, on average, decay to zero faster than the respective singular values, σ_i's [7]. And the representation of $|u_i^T b|$, σ_i, and the respective quotient in a same logarithmic-scale plot is known as a Picard plot [7].

In ill-posed problems, such as ECGI, there may be a point where the data become dominated by errors and the DPC fails. In these cases, to compute a satisfactory solution by means of Tikhonov, the DPC has to be fulfilled [7]. Specifically, the σ_i above the α (useful SVs) must decay to zero slower than the corresponding $|u_i^T b|$ coefficients, ensuring the analytical Tikhonov solution in equation 5 (i.e. $\sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} < \infty$).

2.3. Regularization parameter choice techniques

- **CRESO**

CRESO [5] chooses the parameter value which generates the first local maximum of the difference between the derivatives of the two terms of the Tikhonov function (4),

$$C(\alpha)=\left\{\frac{d}{d(\alpha^2)}(\alpha^2\|a(\alpha)\|^2)-\frac{d}{d(\alpha^2)}\|Ma(\alpha)-b\|^2,\alpha>0\right\} \quad (6)$$

- **L-curve**

The L-curve [7] looks for a parameter providing a good tradeoff between the two terms of (4),

$$L(\alpha)=\{(\|Ma(\alpha)-b\|,\|a(\alpha)\|),\alpha>0\} \quad (7)$$

The regularization parameter can be calculated as the optimal α-value that corresponds to the point on the log-log plot of the L-curve possessing maximum curvature [7].

- **U-curve**

The U-curve method [10] is the plot of the sum of the inverse of the two terms of (4), on a log-log scale

$$U(\alpha)=\left\{\frac{1}{\|Ma(\alpha)-b\|^2}+\frac{1}{\|a(\alpha)\|^2},\alpha\in\left(\sigma_r^{2/3},\sigma_0^{2/3}\right)\right\} \quad (8)$$

being $r=\min(2N_T,N_{S+1})$ and $\sigma_0\geq\sigma_1\geq\cdots\geq\sigma_r>0$. The optimum parameter is the value for which the U-curve has a minimum, i.e. where the two terms of (4) are close. This minimum exists always in the interval $\left(\sigma_r^{2/3},\sigma_0^{2/3}\right)$[10].

- **ADPC: A new regularization parameter choice method based on DPC**

1. We calculate the SVD of M, to obtain the left singular vectors (u_i) and the singular values (σ_i).
2. For each instant of time, $t_k\,(ms)$, we compute $\log(|u_i^T b_{t_k}|)$ and $\log(|u_i^T b_{t_k}|\,/\,\sigma_i)$ and we fit them by two polynomials of degree from 5 to 7 $\left(p\left(i,\log(|u_i^T b_{t_k}|)\right)_{t_k}$ and $q\left(i,\log(|u_i^T b_{t_k}|)\right)_{t_k}$, with $k=1,\cdots,N_t)$.
3. For each pair of polynomials at each instant of time t_k, we find: $\alpha_{t_k}=\sigma_{max\{i\}}\ (\sigma_0\geq\sigma_1\geq\cdots\geq\sigma_r>0)$, such that DPC is fulfilled.
4. $\alpha=\text{median}(\alpha_{t_k})$.

2.4. Experimental and in-silico data used from EDGAR

In this work, eight datasets from EDGAR [12] were used to evaluate the feasibility of U-curve and ADPC, and compare their solutions with the CRESO and L-curve ones.

In the experimental data used, the body surface potentials and epicardial potentials were recorded simultaneously. The data used was: i) A sinus beat and a paced beat from a canine experiment [13]. iii) A sinus beat and a paced beat from a pig experiment [14]. And iii) A control and three myocardial ischemia from a canine experiment [15].

2.5. Validation of the results

- **Regularization parameter choices methods**

We computed the potentials on the epicardium for the different regularization values. Subsequently, correlations coefficients (CC) and root mean square errors (RMSE) were computed through time as in [5].

3. Results

In figure 1, we show the reconstructed epicardial potentials along the time (in a random epicardial location) for the different regularization parameters against the reference epicardial signals in each case.

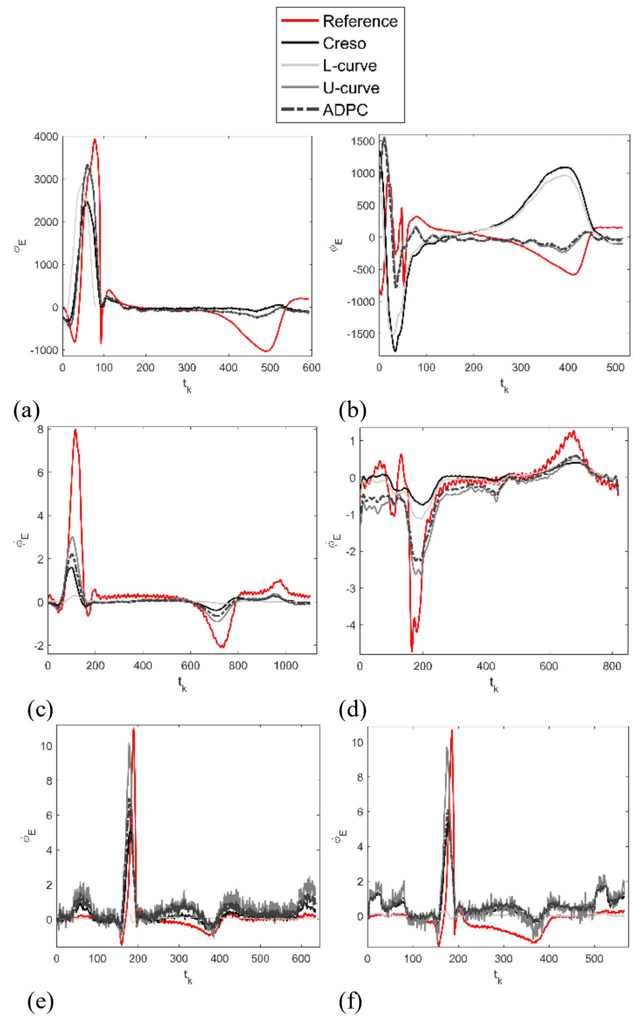

Figure 1. Experiments: (a) Sinus beat canine , (b) Paced beat canine, (c) Sinus beat pig experiment (note L-curve fails), (d) Paced beat pig experiment. (e) Control canine dataset (note L-curve fails). (f) Myocardial ischemia canine experiment (note L-curve fails).

In table 1 we show the differences of the different reconstructions against the classical CRESO one (in %) of the RMSEs and CCs for the potentials along the time. (Higher CCs and lower RMSEs indicates more accuracy).

Table 1. Differences in % of the RMSEs and CCs (Median [Q1, Q3]) of the reconstructed potentials along the time for the different regularization parameters (L-curve, U-curve, ADPC) against the reconstructed potentials by using the CRESO parameter.

Parameter choice	CCs differences (%)	RMSEs differences (%)
L-curve	-0.3 [0.1,-4]	3 [2,1.5]
U-curve	4 [4 ,2.6]	-0.5 [0.1,1]
ADPC	5 [5,3.4]	-2.5 [-1.7,0]

6. Discussion and conclusions

Two new methods are introduced to calculate the regularization parameter of the two-norm Tikhonov method when using MFS for ECGI: The U-curve (a method never used before in cardiac applications) and the ADPC (a new automatic method based on DPC).

The results showed that L-curve performs worse than CRESO for the datasets used in this work (see the plots c, e, and f in figure 1 and results in table 1). However, U-curve and ADPC improved the CCs and the RMSEs of the reconstructed potentials against the provided ones by using CRESO parameter (see table 1).

After the results, it seems that ADPC is the method that provides the most accurate results (+5% CCs and -2.5% RMSEs). Though, U-curve provided better CCs for the paced beat canine datasets than ADPC.

While we anticipate that U-curve is computationally cheaper than L-curve (because it provides a prior interval where the minimum of the U-curve, i.e the optimal parameter can be reach), we need further study in order to compare the computational burden of each method.

Since the behavior of the SVs of ECGI MFS problem (decaying slower for the higher SVs and faster for the lower ones), and the fact that ADPC parameter choice is based on the necessary DPC, ADPC provides a suitable regularization parameter. Nevertheless, it is well known that parameter choice methods is very problem-dependent [7], so if the method wants to be used with other numerical problems such as BEM, we recommend to repeat this study before further conclusions.

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